The role of mixed discounting in risk-averse sequential decision-making

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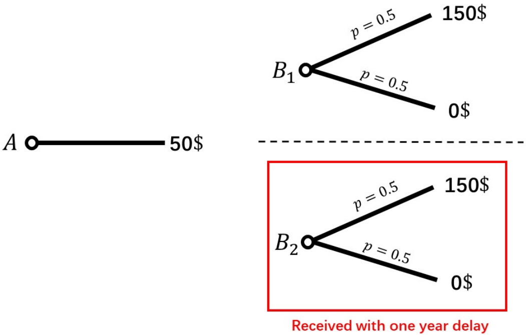
The Role of Discount Factor

- Optimization: Finite value;
- Finance: Prevailing interest rate – Compute the present value of future cash flows values;
- Behavioral Economics: Time preference/Impulsivity: Favoring present vs. future;

A Lottery Choice Example



- Use power utility model $u(x) = x^\alpha$ where $\alpha = 0.9$.
- Plan A and B_1 have expected utility 33.81 and 45.44, respectively.
- An annual discount factor $\gamma \in (0, 1)$ is introduced. The "discounted" utility is $u(\gamma x) = (\gamma x)^\alpha$. If $\gamma \in [0.72, 1)$, $A \preceq B_2$; otherwise, $A \succeq B_2$.

Basic Models

- Given two potential expenses z_0 and Z_1, with z_0 an immediate deterministic amount while Z_1 is a random one received at time T, the risk-averse exponentially discounted preference is:

$$\text{(D-model)} \qquad \mathcal{R}(z_0, Z_1, T) := \rho(z_0 + \gamma^T Z_1),$$

  where ρ is a risk measure (RM).

- The representation reduces to the well known expected total discounted cost when $\rho(X) := \mathbb{E}[X]$:

$$\text{(E-model)} \qquad \mathcal{R}(z_0, Z_1, T) := \mathbb{E}[z_0 + \gamma^T Z_1] = z_0 + \gamma^T \mathbb{E}[Z_1]$$
$$= \mathbb{E}[z_0 + \mathbf{1}\{\tau \geq T\} Z_1],$$

  where τ is a random interruption time which follows an exponential distribution with mean $-1/\ln(\gamma)$.

[Shwartz 2001, Ermoliev 2010]

Basic Models (Con)

- The random interruption equivalence does not carry through for more general forms of RMs. For instance,

$$\begin{aligned}\mathbb{E}[u(z_0 + \{\tau \geq T\}Z_1)] &= (1-\gamma^T)u(z_0) + \gamma^T[u(z_0 + Z_1)] \\ &\neq [u(z_0 + \gamma^T Z_1)].\end{aligned}$$

  unless $u(\cdot)$ is linear.

- The proposed random interruption model ("RI-model"):

$$\text{(RI-model)} \qquad \mathcal{R}(z_0, Z_1, T) := \rho(z_0 + \mathbf{1}\{\tau \geq T\}Z_1),$$

  and more generally a mixture model ("M-model"),

$$\text{(M-model)} \qquad \mathcal{R}(z_0, Z_1, T) := \rho(z_0 + \mathbf{1}\{\tau \geq T\}\gamma_d^T Z_1),$$

  where τ follows an exponential distribution with mean $-1/\ln(\gamma_r)$.

Empirical Study Supporting the M-model

- Existing field experiment results [Lopez-Guzman 2018] (Acknowledgment: Grateful for the authors providing field experiment data for this research !);
- Subjects were asked to undertake two sessions: Risk Attitude (RA) task and Inter-Temporal Choice (ITC) task;
- Logistic regression;

1. The monthly discounting ranged from 2.9% to 99.5% of the reward, with a median of 60%. From a purely financial point of view, note that a monthly discounting of 2.9% is already equivalent to assuming a yearly interest rate of nearly 42%. In comparison, the average credit card rate in the US over the period of 2014-2018 was below 17%.

$$\text{Discounting} = 1 - \frac{1}{(1 + \text{Interest rate})^{\text{Period}}}$$

2. RI-model achieves a higher goodness-of-fit performance than the D-model, but whether the D-model can indeed safely be rejected, is left for investigation.

Empirical Study Supporting the M-model (Con)

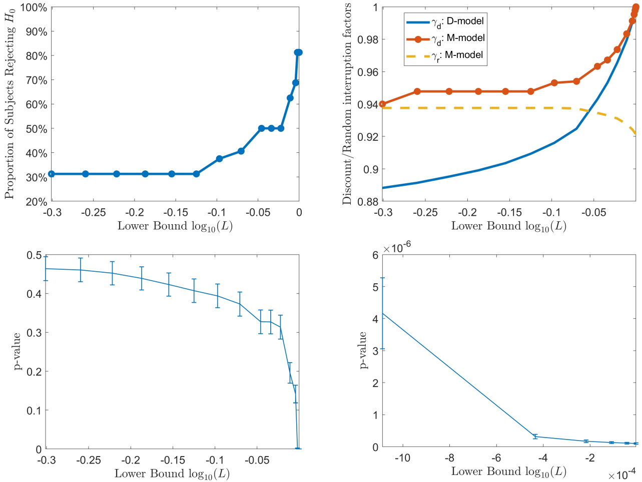
- New features in our analyses:
  1. Use Entropic risk measure: $\rho(X) := (1/\beta)\ln(\mathbb{E}[\exp(\beta X)])$, with $\beta > 0$ as the risk aversion parameter; 32 participants are shown to be risk-averse by RA task.
  2. Credit card assumption: Subjects have access to credit with a daily interest rate less than $(1/L - 1)$ for some $L > 0$ and cannot create value directly from this credit instrument.

     $\Rightarrow$ We necessarily have that $\gamma_d \geq L$ (constrained case).
  3. Conduct likelihood ratio tests for D-model and M-model, under both constrained and unconstrained cases;

- Unconstrained case: Estimated discount factor γ_d is consistent with the findings in [Lopez-Guzman 2018]. For 10 out of the 32 participants, the D-model can safely be rejected in favor of the M-model.

Empirical Study Supporting the M-model (Con)

- Constrained case: 26 subjects (81% of the population) eventually reject D-model as L approaches 1.



Risk Preference Mappings

- Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathfrak{F} := (\mathcal{F}_t)_{t\in\mathbb{N}}$ be a filtration, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_t \subset \mathcal{F}_{t+1}$ for all $t \in \mathbb{N}$. Consider for all $t \in \mathbb{N}$, a set of random liability $\mathcal{Z}_t \subseteq L_p(\Omega, \mathcal{F}_t, P)$ for some $p \in [1, \infty]$. Consider cash flows $Z_{\mathbb{N}} := (Z_t)_{t\in\mathbb{N}}$ with $Z_t \in \mathcal{Z}_t$ for all $t \geq 0$, adapted to filtration $\mathfrak{F}$. Define a preference mapping $\mathcal{R} : \mathcal{Z}_{\mathbb{N}} \to \mathbb{R}$ that is both monotone, convex, and recursive.

$$\text{(D-model)} \qquad \mathcal{R}_D(Z_{\mathbb{N}}) := \limsup_{T\to\infty} \rho\Big(\sum_{t=0}^{T} \gamma^t Z_t\Big).$$

$$\text{(M-model)} \qquad \mathcal{R}_M(Z_{\mathbb{N}}) := \limsup_{T\to\infty} \rho\Big(\sum_{t=0}^{\min\{\boldsymbol{\tau}, T\}} \gamma_d^t Z_t\Big),$$

where $\mathbb{P}[\boldsymbol{\tau} = t] = (1-\gamma_r)\gamma_r^t$ and $\mathbb{P}[\boldsymbol{\tau} \geq t] = \gamma_r^t$, and $1\{\boldsymbol{\tau} \geq t\}$ is adapted to the filtration $\mathfrak{F}$.

Risk Preference Mappings (Con)

Definition 1 (Recursive preference mapping)

The preference mapping $\mathcal{R}(\cdot)$ is recursive if there exists a preference system $\{\bar{\mathcal{R}}_t\}_{t\in\mathbb{N}}$, with each $\bar{\mathcal{R}}_t : \mathcal{Z}_t \times \mathcal{Z}_{t+1} \to \mathcal{Z}_t$ such that, for all $Z_{\mathbb{N}} \in \mathcal{Z}_{\mathbb{N}}$:

$$\mathcal{R}(Z_{\mathbb{N}}) = \limsup_{T\to\infty} \bar{R}_0(Z_0,\ \bar{R}_1(Z_1, \ldots, \bar{R}_{T-1}(Z_{T-1}, Z_T) \ldots)).$$

Definition 2 (Recursive risk measure)

The risk measure $\rho : \mathcal{L}_p(\Omega, \mathcal{F}, P) \to \mathbb{R}$ is a "recursive risk measure" if there exists a set of risk measures $\{\rho_t\}_{t=0}^{\infty}$ such that

$$\rho(X) = \limsup_{T\to\infty} \rho_0(\rho_1 \cdots \rho_{T-2}(\rho_{T-1}(\mathbb{E}[X|\mathcal{F}_T])) \cdots),$$

where each $\rho_t : \mathcal{L}_p(\Omega, \mathcal{F}_{t+1}, P) \to \mathcal{L}_p(\Omega, \mathcal{F}_t, P)$ is a conditional risk mapping [Ruszczynski 2006].

Risk Preference Mappings (Con)

Theorem 3 (Recursive Formulation of D-model)

D-model satisfies $\bar{\mathcal{R}}_t(Z_t, Z_{t+1}) := Z_t + \gamma^{-t}\rho_t(\gamma^{t+1}Z_{t+1})$*, which reduces to* $\bar{\mathcal{R}}_t(Z_t, Z_{t+1}) := Z_t + \gamma\rho_t(Z_{t+1})$*, when the conditional risk mappings are coherent. In the latter case, we have that:*

$$\mathcal{R}_D(Z_{\mathbb{N}}) = \limsup_{T\to\infty} Z_0 + \gamma\rho_0(Z_1 + \gamma\rho_1(Z_2 + \cdots + \gamma\rho_{T-1}(Z_T)\cdots)). \quad (1)$$

Risk Preference Mappings (Con)

Theorem 4 (Recursive Formulation of M-model)

M-model satisfies $\bar{\mathcal{R}}_t(Z_t, Z_{t+1}) := Z_t + \gamma_d^{-t}\rho_t(0 \oplus_{\gamma_r} \gamma_d^{t+1} Z_{t+1})$*, which reduces to* $\bar{\mathcal{R}}_t(Z_t, Z_{t+1}) := Z_t + \gamma_d \rho_t(0 \oplus_{\gamma_r} Z_{t+1})$*, when the conditional risk mappings are coherent. In the latter case, we have:*

$$\begin{aligned}&\mathcal{R}_M(Z_{\mathbb{N}})\\ =&\limsup_{T\to\infty} Z_0 + \gamma_d \rho_0(0 \oplus_{\gamma_r} (Z_1 + \gamma_d \rho_1(0 \oplus_{\gamma_r} (Z_2 + \cdots + \gamma_d \rho_{T-1}(0 \oplus_{\gamma_r} Z_T)\cdots)))).\end{aligned} \tag{2}$$

An Optimal Stopping Problem Example

- Consider a stochastic process $A_{\mathbb{N}} := (A_t)_{t\in\mathbb{N}}$ of periodical cost (negative of payment) opportunities with each $A_t \in \mathcal{A} \subset \Re_-$. DM must identify when to accept the payment using a controlled stop time (CST) process $\mathbf{s}$ such that $\mathbf{1}\{\mathbf{s} \geq t\}$ is adapted to the natural filtration of the $A_{\mathbb{N}}$ process.
- Given a CST strategy $\mathbf{s}$, the cost flow produced takes the form: $Z_t(\mathbf{s}) := A_{\mathbf{s}}\mathbf{1}\{t = \mathbf{s}\}$ for all $t \in \mathbb{N}$. Letting $\mathcal{S}$ denote the set of all eligible CST strategies, we have the problem:

$$\min_{\mathbf{s}\in\mathcal{S}} \mathcal{R}(Z_{\mathbb{N}}(\mathbf{s})).$$

Assume that $\mathcal{A}$ is bounded (i.e. $\mathcal{A} \subseteq [-\bar{A}, \bar{A}]$) and each A_t are i.i.d.

An Optimal Stopping Problem Example (Con)

Proposition 1

When $\gamma_d < 1$ in general and when $\gamma_r < 1$ if the conditional risk mappings are either comonotone additive or the entropic risk measure and 2) $\mathcal{A} \subset (-\infty, 0]$, the infinite-horizon problem can be approximated to any level of precision using a finite-horizon one.

- Dynamic programming formulation for T horizon problem:

$$V_t^*(A_t) = \min\left\{A_t,\ \gamma_d^{-t}\rho_t(0 \oplus_{\gamma_r} \gamma_d^{t+1} V_{t+1}^*(A_{t+1}))\right\},$$

with $V_T^*(A_T) = \min\{0, A_T\}$. An optimal policy is

$$s_T^* = \inf\{t : V_t^*(A_t) = A_t\} = \inf\{t : A_t \leq R_t\}.$$

- Equivalently, we could solve $R_t = \gamma_d^{-t}\rho(0 \oplus_{\gamma_r} \gamma_d^{t+1} \min\{R_{t+1}, \mathbf{a}\})$, with $R_T = 0$, where $\mathbf{a}$ follows certain distribution on $[-\bar{A}, 0]$.

An Optimal Stopping Problem Example (Con)

- We observe that,

$$\begin{aligned} R_t =& \gamma_d^{-t} \rho(0 \oplus_{\gamma_r} \gamma_d^{t+1} \min\{R_{t+1}, \mathbf{a}\}) \\ =& \frac{1}{\beta\gamma_d^t} \log(\mathbb{E} \exp(0 \oplus_{\gamma_r} (\beta\gamma_d^t)\gamma_d \min\{R_{t+1}, \mathbf{a}\})), \end{aligned}$$

  when $t \to \infty$ and $\gamma_d < 1$, the risk parameter in the conditional entropic risk measure $\beta\gamma_d^t \to 0$ is dissolved, such that we are approaching for solving the risk-neutral equation.

- Under $\mathbf{a} \sim U[-\bar{A}, 0]$, Setting the truncation

$$R_T = \bar{A}(\gamma_d\gamma_r)^{-1}(\sqrt{1 - (\gamma_d\gamma_r)^2} - 1),$$

  instead of $R_T = 0$, yields a better approximation [Hau 2022].

The Properties of D- and M-model

- Finiteness

Proposition 2

Given that $Z_t \in [-B, \ B]$ for all $t \geq 0$, $B \geq 0$, and ρ is a normalized nested risk measure with law-invariant conditional risk mappings, then the M-model is finite if either of the following hold:

(i) $\gamma_d < 1$,

(ii) $\gamma_r < 1$ *and* $Z_t \leq 0$ *for all* $t \geq 0$,

(iii) when $\gamma_r < 1$ *and the nested risk measure is composed of utility-based shortfall conditional risk mappings [Follmer 2002] with loss functions with subdifferentials bounded in a strictly positive interval,*

(iv) when $\gamma_r < 1$ *and the nested risk measure is composed of optimized certainty equivalent [Ben-Tal 1986] conditional risk mappings that employ a surjective and stationary loss function.*

The Properties of D- and M-model (Con)

- Interchangeability

Proposition 3

Given that $Z_t \in [-B, 0]$ *for all* $t \geq 0$, *and* ρ *is a normalized recursive risk measure with subjective conditional measures, then RI- and M- model can be reformulated as D-model with adjusted lower risk measure and lower discount factor.*

- * Counter-example: Entropic risk measure

The Properties of D- and M-model (Con)

- Ordering

Proposition 4

Under a recursive risk measure composed of law-invariant conditional measures, and if $\gamma_d < 1$ and bounded $X_t \geq 0$ for all $t \geq 0$, M-model will lower bound D-model with $\gamma = \gamma_d$.

Definition 1

The conditional mapping ρ_t is "mixture concave" if
$\forall\, \gamma \in [0,\, 1], \forall\, X, Y \in \mathcal{L}_p(\Omega, \mathcal{F}_{t+1}, P)$, $\rho_t(X \oplus_\gamma Y) \geq (1-\gamma)\rho_t(X) + \gamma\rho_t(Y)$ *a.s.*

Proposition 5

Under a recursive risk measure composed of law-invariant and "mixture concave" conditional measures (not necessarily coherent), M-model will upper bound D-model with $\gamma = \gamma_d\gamma_r$ and $\bar{\rho}_t(X) := \gamma_r^{t+1}\rho_t(\gamma_r^{-(t+1)}X)$ as the conditional risk mapping. The latter reduce to $\bar{\rho}_t(X) := \rho_t(X)$ in the case that ρ is coherent.

The Properties of D- and M-model (Con)

- Examples of "mixture concave" convex risk measure:
  1. [Delage 2019] Spectral risk measure, Mean variance, Mean standard deviation, Mean (semi-) deviation from target and Mean weighted mean deviation from quantile.
  2. Optimized certainty equivalent:

$$
\begin{aligned}
&\rho(X \oplus_\gamma Y) \\
=&\inf_{t\in\mathbb{R}}\{t + \mathbb{E}[\ell(X \oplus_\gamma Y - t)]\} \\
=&\inf_{t\in\mathbb{R}}\{t + (1-\gamma)\mathbb{E}[\ell(X - t)] + \gamma\mathbb{E}[\ell(Y - t)]\} \\
\geq&(1-\gamma)\inf_{t\in\mathbb{R}}\{t + \mathbb{E}[\ell(X - t)]\} + \gamma\inf_{t\in\mathbb{R}}\{t + \mathbb{E}[\ell(Y - t)]\} \\
=&(1-\gamma)\rho(X) + \gamma\rho(Y).
\end{aligned}
$$

  3. [Postek 2016] Other minimum of affine function with expected loss

- *Counter-example: Utility-based shortfall with piecewise loss function

Numerical Experiments

- $\bar{A} = 10$; $T = \lceil 1/(1-\gamma_d) \rceil$ for D-model and $T = \lceil 1/(1-\gamma_d \gamma_r) \rceil$ for M-model;

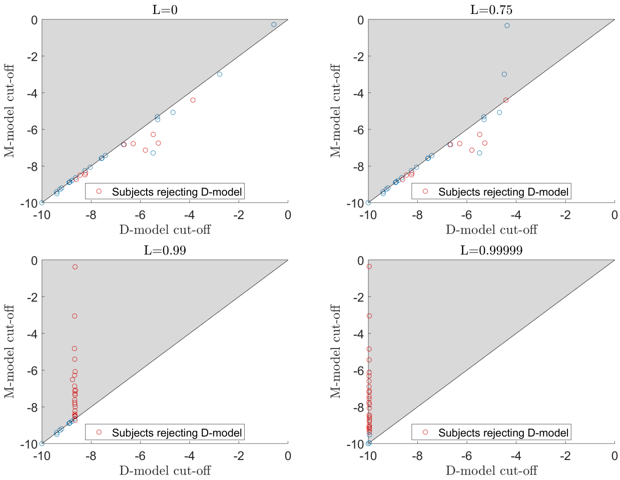


Figure: Distribution of D-model and M-model cut-off values pair

Numerical Experiments (Con)

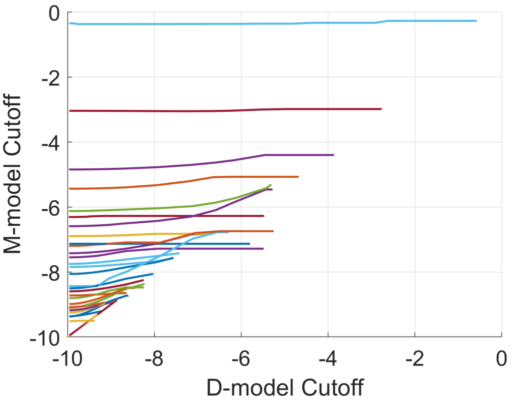


Figure: Cutoffs for D- and M-model as a function of L (Each curve represents a participant of the study)

Numerical Experiments (Con)

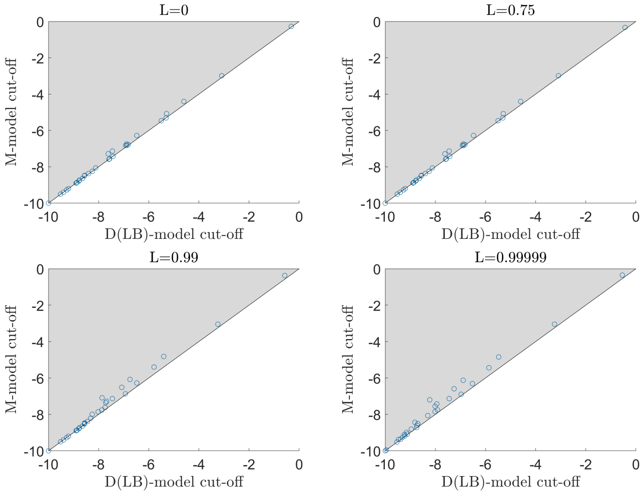
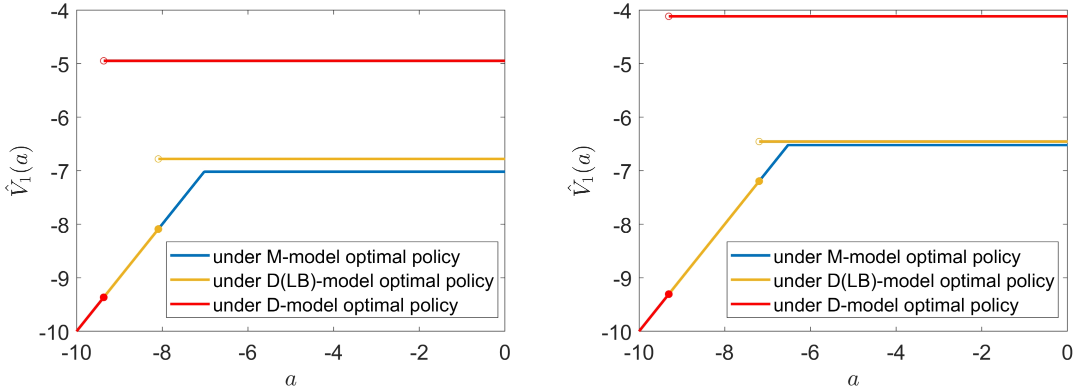


Figure: Distribution of D(LB)-model and M-model cut-off values pair

Numerical Experiments (Con)



(a) Unconstrained participant (b) Constrained participant

Figure: Conditional first-stage risk of running the D-model, D(LB)-model, and M-model policy when true model is an M-model

Numerical Experiments (Con)

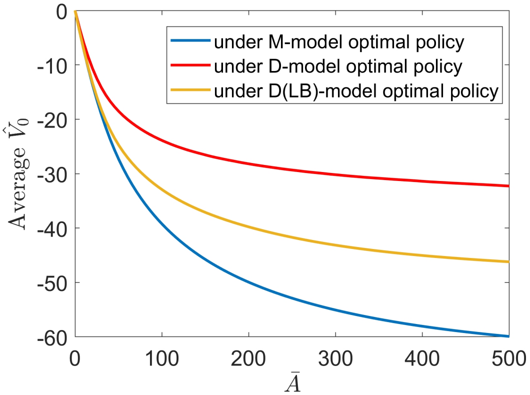


Figure: Average $\hat{V}_0$ for rejected subjects of running D-model, D(LB)-model and M-model policy when the true model is an M-model

Conclusion

1. Present evidences through field experiment data that decision makers are better represented by a propsoed M-model under certain assumptions;
2. Extend the “risk preference mapping” framework [Pichler 2022] to an infinite horizon setting and show how it can be reduced to M-model;
3. Numerical experiments on a simple optimal stopping problem, validate the effects of the proposed model;
4. Address the relationships between D-, and M-model (RI-model);

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Thank you !

Q & A

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